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Note

A Note on Rational Approximation to $(1-x)^\alpha$

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Let the set of polynomials of degree at most n and having nonnegative real coefficients be denoted by Π_n^+. If $\|f\|$ means $\sup_{[0,1]} |f(x)|$, let us write

$$H_n[f] = \inf_{p,q \in \Pi_n^+} \|f - p/q\|.$$

It has been shown recently that the exact order of $H_n[(1-x)^{1/2}]$ is $n^{-1/2}$. This is a consequence of the result

$$\|(1-x)^{1/2} - (p(x)/q(x))\| \geqslant \tfrac{1}{4} n^{-1/2} \qquad (p, q \in \Pi_n^+, n \geqslant 12) \tag{1}$$

due to Reddy [1] and the result

$$\|(1-x)^{1/2} - T_n^{-1}(x)\| \leqslant (\sqrt{\pi}/2)\, n^{-1/2} \qquad (n \geqslant 1) \tag{2}$$

due to Bundschuh [2]. Here $T_n(x)$ denotes the nth Taylor polynomial of $(1-x)^{-1/2}$ which clearly belongs to Π_n^+.

In the present note we generalize result (2) to the case of the function $(1-x)^\alpha$ $(0 < \alpha \leqslant 1)$. The method used is quite different from that used by Bundschuh and treats all α in the range $0 < \alpha \leqslant 1$ simultaneously. Our result is the Theorem stated below.

The proof of Reddy's result (1) can be extended with little change to cover each value of α, $0 < \alpha \leqslant 1$, whence it is found that

$$\|(1-x)^\alpha - (p(x)/q(x))\| \geqslant \tfrac{1}{4} n^{-\alpha} \qquad (p, q \in \Pi_n^+, n \geqslant 12).$$

This, combined with our Theorem, gives the exact order of $H_n[(1-x)^\alpha]$ as $n^{-\alpha}$ $(0 < \alpha \leqslant 1)$.

Our main result is

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THEOREM. *If $T_n(\alpha, x)$ is the nth Taylor polynomial of $(1-x)^{-\alpha}$ $(0<\alpha\leqslant 1)$ (which belongs to Π_n^+), then*

$$\|(1-x)^{\alpha}-T_n^{-1}(\alpha, x)\| \leqslant K\Gamma(\alpha)\, n^{-\alpha} \qquad (n \geqslant 1),$$

where K is a constant independent of both n and α.

Proof. The function $(1-x)^{-\alpha}$ is the unique solution of the differential equation

$$(1-x)\, y' - \alpha y = 0 \qquad \text{with} \quad y(0)=1.$$

Written in series form the solution is

$$(1-x)^{-\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)}\, x^k \qquad (|x|<1)$$

and so

$$T_n(\alpha, x) = \sum_{k=0}^{n} \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)}\, x^k.$$

It is then easy to see that $T_n(\alpha, x)$ is the unique solution of the differential equation

$$(1-x)\, y' - \alpha y = -\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha)\, n!}\, x^n \qquad \text{with} \quad y(0)=1.$$

Solving this by a standard technique we find that

$$T_n(\alpha, x) = (1-x)^{-\alpha} - \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha)}\, (1-x)^{-\alpha} \int_0^x t^n (1-t)^{\alpha-1}\, dt.$$

Dividing by $(1-x)^{-\alpha}\, T_n(\alpha, x)$, we get

$$(1-x)^{\alpha} = \frac{1}{T_n(\alpha, x)} - \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha)}\, \frac{1}{T_n(\alpha, x)} \int_0^x t^n (1-t)^{\alpha-1}\, dt. \qquad (3)$$

We now examine the last term. We have

$$\begin{aligned} 0 \leqslant \int_0^x t^n (1-t)^{\alpha-1}\, dt &= x^{n+1} \int_0^1 t^n (1-xt)^{\alpha-1}\, dt \\ &\leqslant x^{n+1} \int_0^1 t^n (1-t)^{\alpha-1}\, dt \\ &= x^{n+1}\, \frac{\Gamma(n+1)\,\Gamma(\alpha)}{\Gamma(n+\alpha+1)}. \end{aligned}$$

Also

$$T_n(\alpha, x) = \sum_{k=0}^{n} \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{1}{k!} x^k \geqslant \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, n!} \sum_{k=0}^{n} x^k$$

because the coefficients are positive and decrease with k. We note that

$$\frac{x^{n+1}}{T_n(\alpha, x)} \leqslant \frac{x^{n+1}}{\sum_{k=0}^{n} x^k} \frac{\Gamma(\alpha)\, n!}{\Gamma(n+\alpha)} \leqslant \frac{1}{n+1} \frac{\Gamma(\alpha)\, n!}{\Gamma(n+\alpha)}.$$

The last step here depends on the observation that $x^{n+1}(\sum_{k=0}^{n} x^k)^{-1}$ is an increasing function in $0 \leqslant x \leqslant 1$. Applying these results to (3), we now see that

$$0 \leqslant \frac{1}{T_n(\alpha, x)} - (1-x)^\alpha \leqslant \frac{n+\alpha}{n+1} \frac{\Gamma(\alpha)\, \Gamma(n+1)}{\Gamma(n+\alpha+1)}.$$

Since $n^\alpha(\Gamma(n)/\Gamma(n+\alpha)) \to 1$ as $n \to \infty$, uniformly with respect to α in $0 < \alpha \leqslant 1$, the proof of the theorem is complete.

References

1. A. R. Reddy, A note on rational approximation to $(1-x)^{1/2}$, *J. Approx. Theory* **25** (1979), 31–33.
2. P. Bundschuh, A remark on Reddy's paper on the rational approximation of $(1-x)^{1/2}$, *J. Approx. Theory* **32** (1981), 167–169.

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